Learning Continuum for Addition and Subtraction

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- Jumps in level are made by students, not by teachers or the curriculum. Teachers can support students by having students work on problems that stretch them but do not overwhelm. Have students explain their reasoning and show them alternative ways of solving problems.
- The levels of sophistication is students' reasoning follows this general progression:
    1. Students use physical materials to model the action or relationships described in the problems and use counting to solve the problem.
    2. Students use counting only strategies; first by ones and then by place-value parts.
    3. Students then combine and decompose numbers by place-value parts without counting.
    4. Students understand and use symbolic algorithms.

- Once students understand numbers, they can progress to developing conceptual understanding of physical situations that give rise to addition and subtraction. There are 3 basic situations:
    1. Addition – students understand addition in terms of physically joining sets of objects. A somewhat more abstract situation occurs when two sets of objects are mentally joined rather than physically joined.
    2. Subtraction – initially students understand subtraction in terms of physically taking away one set of object from another.
    3. Comparison – "Tommy has 8 black marbles and Susie has 5 grey marbles. How many more marbles does Tommy have?" These problems can be solved using either addition or subtraction and serve to introduce the idea that addition and subtraction are inverse operations.

- It is extremely important that students develop an understanding of the inverse relationship between addition and subtraction.
- Implementing reasoning strictly verbally is more sophisticated than implementing it concretely or pictorially.
- When using place value blocks the prominent feature should be their numeric value not their shape. Thus when we are verbally describing them use the term "ten" or "ten block" not "strip" or "rod".

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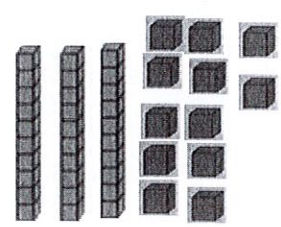
Developmental Level	**Addition and Subtraction Description**
Developmental Level 0	**Student does not understand addition and subtraction situations. (No use of algorithms)** Task: Johnny has 5 pencils. His teacher gives him 2 more pencils. How many pencils does Johnny have? Response: Johnny has 5 pencils.
Developmental Level 1	**Student adds or subtracts numbers as collections of ones. (no skip-counting by place value) No use or rote use of algorithms.**
1.1	Student counts all. Task: What is 3 + 8 Response: Student makes one set of 3 cubes and a second set of 8 cubes then Counts them, " 1, 2, 3, 4, ... 11"  • Some students may be able to count using imagined objects. To assist in this transition some students may need to see the objects first and then they can be hidden. • Counting using fingers is more difficult than counting using objects because students must produce and operate on the representations they have created • Counting using counting words is more difficult as students must keep track of how many counting words they have used. They may use their fingers.
1.2	Student counts on or counts down. Task: 3 + 8 Response: (Putting up 8 fingers one at a time) 4, 5, 6, 7, 8, 9, 10, 11. • Requires a major jump in abstraction as student must recognize that 3 exists even when they are not involved in counting it. • This is difficult for students as they must coordinate two number sequences simultaneously – the counting sequence and the number of items to be counted. (ex. 4 through 1 which corresponds with 1 through 8). • A common mistake is for the student to start at 3 and have this quantity correspond to 1 of the 8 items being counted.

1.3	Student recalls or derives a fact using number properties. Task: 3 + 8 Response: 11. (How did you get that?) I know it because it is a basic fact. Commutative Property of Addition: 3 + 8 = 8 + 3 Associate Property of Addition: 3 + 8 = (1+2) + 8 = 1 + (2+8) = 1 + 10 Inverse Relationship: 3 + 8 = 11 so 11 – 8 = 3 Fact Families are formed through these relationships. 3 + 8 = 11 8 + 3 = 11 11 – 3 = 8 11 – 8 = 3
1.4	Student operate on tens and ones separately as ones. (Students begin to treat the ones and tens digits differently but do not understand the relationship between tens and ones. Students are using rote to perform operation and represent numbers visually). Task: 17 + 25 = Response: 1 + 2 = 3 (writes the 3) 7 + 5 = 12. I don't know whether to write the 1 or 2. I have been told it is always the last number. (writes the 2 on her paper). The answer is 32.
Developmental Level 2	**Student adds or subtracts numbers by skip-counting place-value parts. No use or use of algorithms with weak or no connection between place value and algorithms.**
2.1	Student counts by tens and ones separately. Task: In a box, there are 17 green apples and 25 red apples. How many apples are in the box? Response: Student represents each number. She moves all the ten blocks and ones blocks together. 10, 20, 30, 31, 32, … 41, 42. 
2.2	Student counts by tens in mid-decades. Task: 17 + 25 Response: 17, 17, 37, 38, 39, 40, 41, 42

Developmental Levelt3	**Student adds or subtracts by combining or separating place-value parts. (Explicit use of place value in informal multi-digit arithmetic; emerging but incomplete understanding of place value in algorithms.**
3.1	Student uses landmarks. Task: 17 + 25 +3 +22 Response: ______17_____20________________42_
3.2	Student separately adds or subtracts the tens and ones parts of number to or from the other number. Task: 17 + 25 Response: 20 + 25 = 45. But it is 17 so you have to subtract 3. It's 42.
3.3	Student combines or separates tens parts and ones parts and then adds the results. (Counting is not the primary strategy) Task: 17 + 42 Response: 7 plus 5 is 12. 10 plus 20 is 30. So, if you add the 10 from the 12 to 30, you get 40 then take the 2 from 12 and get 42. • Students may use physical materials but not in a way that is consistent with the traditional algorithm. (May not represent the numbers first or may not regroup). These discrepancies are one reason students may have difficulty connecting their thinking with the traditional algorithm. • Using only a verbal approach students at this level always add or subtract the tens without counting, but them might add, subtract, or count ones.
Developmental Level 4	**Student uses and understands expanded addition and subtraction algorithms. (Place-value understanding of expanded algorithms through hundreds).** Task: 17 + 25 Response: Place-Value parts explicitly shown: 10 +7 7 plus 5 is 12 20 + 5 10 plus 20 is 30 30 +12 = 42 30 plus 12 is 42 Place-Value parts explicit in verbal: 17 7 plus 5 is 12 25 10 plus 20 is 30 42 30 plus 12 is 42 • Students operating at this level reliably use the expanded algorithm • Generally students learn addition algorithm before subtraction

	• These are very important since they reduce the cognitive load making it less likely that students will make errors and it also provides the conceptual knowledge for the traditional algorithm. • To understand these algorithms students must understand place value and how to combine tens and ones to make a two digit number.
Developmental Level 5	**Student uses and understands traditional addition and subtraction algorithms. (Place value understanding of traditional algorithms through hundreds.)** Task: 17 + 42 Response: 7 plus 5 is 12 write the 2 below the 5, write 1 above the 1. 1 plus 1 plus 2 is 4. • Place value is dealt with implicitly by the placement of the digits, not in the language, so place values are hidden. • Students can provide a conceptually based explanation of why the steps they perform are valid. • To determine the level of understanding teachers need to ask probing questions such as "Is this really a one?" (when carrying) • Students need to be at a level 3.3 prior to introducing algorithms. Often algorithms are introduced too early. Students may exhibit four different types of understandings of algorithms: 1. Student incorrectly performs the algorithm. Often conceptual errors are very apparent. 2. Student correctly performs the algorithm, but with a clear indication that it is not understood. (Responses incorrectly to probing question) 3. Student correctly performs the algorithm with no indication of understanding or misunderstanding. (Probing questions are needed) 4. Student correctly and meaningfully use the algorithm.

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