

Learning Continuum for Multiplication and Division

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- There are two critical components in the development of a student's reasoning about multiplication and division of whole numbers.
 - The student must understand what the operations mean and recognize when each is appropriate in problem solving.
 - The student must understand and become proficient with strategies for performing computations for these operations.
- Two types of Division can cause some confusion for students:
 - Measurement Division: the total number of objects are given and one is asked how many equivalent groups of a given size can be made from the total.
 - Partitive Division: the total number of objects are given and one is asked how many objects would be in each group if partitioned into a given number of equivalent groups.
- The student begins to develop multiplication by iterating (repeating and accumulating) units. In order to use iterate reasoning students must be able to use counting sequences/skip count.
- The student must learn the inverse relationship between multiplication and division.
- Although reasoning for multiplication develops before division, both can be taught and learned at the same time.
- An important part of development is understanding and using computational algorithms. However, algorithms should not be taught too early in the student's development of reasoning or they are learned by rote and not conceptually.

Developmental Level	Description
Developmental Single Digit Level 0	Student does not understand multiplication or division situations. The student will count individual objects rather than a group of objects. The student may not distribute objects into equal groups.
Developmental Single Digit Level 1	Student multiples or divides numbers by counting objects in groups by ones with no skip counting. Students needs to have the representation and be able to count objects by ones.
1.1	Student counts physical or visual objects by ones. Without the manipulatives, the student is not successful.

1.2	<p>Student correctly counts visualized/imagined objects or counting words by ones.</p> <p>The student would use their fingers to help count groups. For example: How many cubes are in 5 stacks of 3? Student responds with 1,2,3 (then holds up one finger), 4,5,6 (holds up a second finger), ...</p>
1.3	<p>Student uses uncoordinated, incorrect skip-counting.</p> <p>Student will skip-count incorrectly because they cannot simultaneously keep track of two counting schemes – one for the groups and one for the total number of objects in the group.</p>
Developmental Single Digit Level 2	<p>Student multiples/divides numbers by repeated addition/subtraction or skip-counting.</p> <p>These are crucial skills to develop. 2.1 and 2.3 are the most important goals for achieving this level.</p>
2.1	<p>Student uses repeated addition or subtraction, or skip counts and counts by ones.</p> <p>The student would be familiar with the beginning sequence of skip-counting, but needs support knowing the pattern for later values. The student would begin skip-counting and then end by counting by ones.</p>
2.2	<p>Student decomposes a number into parts and skip-counts those parts. This is similar to 2.1, but the student intentionally decomposes the number into parts – which is a higher form of math reasoning.</p>
2.3	<p>Student skip-counts all multiples in the skip-count sequence without decomposing numbers into parts.</p> <p>The student is able to keep track of groups/total number of objects and is reliable with the number of skip-counts.</p>
2.4	<p>Student skip-counts a group of skip-counts.</p> <p>$9 \times 4 = 4, 8, 12$; that's 3. Three more 4s would be 24; Three more would be 36.</p>

Developmental Single Digit Level 3	<p>Student multiplies or divides numbers by recalling facts or by using properties to derive answers from known facts with no counting or skip-counting.</p> <p>Facts should be <i>meaningful</i> – student can conceptualize the number rather than know it by rote.</p>
3.1	Student directly recalls basic multiplication or division facts.
3.2	<p>Student uses number properties to derive answers from known facts.</p> <ul style="list-style-type: none"> • Commutative Property of Multiplication • Inverse Relationship between Multiplication and Division • Associative Property of Multiplication • Division/Multiplication Property • Distributive Property for Multiplication • Distributive Property for Division
Developmental Multi-Digit Level 1	Student multiplies or divides numbers by counting objects in groups by ones with no skip-counting.
1.1	<p>Student counts physical or pictorial objects by ones (but not using place-value blocks). If physical/visual materials are not used, the student make mistakes in counting.</p> <p>For example: 6×23 = student has to draw 6 circles with 23 dots in each and then counts by ones.</p>
1.2	<p>Student correctly counts groups of counting words by ones.</p> <p>The student would use his fingers to help count out groups.</p>
1.3	<p>Student use uncoordinated and/or incorrect skip-counting.</p> <p>The skip-counting of multi-digit numbers is too difficult to keep track with the groups and total objects in each group.</p>
1.4	<p>Student iterates multi-digit numbers with place value blocks, but still operates on the blocks as ones</p> <p>The student would use manipulatives to solve, but may not fully understand the relationship between the 10s and the 1s and only counts by ones. For example: The student recognize 6 tens as 60 but may still call it 6. I have 6 tens and 3 ones.</p>
Developmental Multi-Digit	Student multiplies or divides numbers by repeated addition/subtraction or skip-counting.

Level 2	Note/Goal: Students in Level 2 will have had success with multiplying and dividing by using repeated addition/subtraction and skip-counting, but need to be working towards strategies that are more efficient and not so tedious.
2.1	Student uses repeated addition or subtraction or some counting by ones.
2.2	Student skip-counts nonplace-value parts of a group of skip-counts. The student decomposes a number into smaller parts that they can skip-count more easily but they do not use place value parts. The student will group a number of skip counts together to shorten the counting.
2.3	Student skip-counts all numbers in the skip-count sequence without decomposing numbers into parts.
2.4	Student skip-counts by place-value parts. This is done with and without physical/visual materials. For example: $26 \times 9 = 20, 40, 60, 80, 100, 120, 140, 160, 180$ $= 9, 18, 27, 36, 45, 54$ $= 180 + 54$
Developmental Multi-Digit Level 3	Student multiplies or divides numbers by using properties to combine or separate parts with no counting or skip-counting.
3.1	Student use known multiplication or division facts and number properties to derive answers but does not use the distributive property with place-value decompositions.
3.2	2 Digit X 1 Digit: Student use the distributive property to decompose numbers by place value into two partial products.
3.3	2 Digit X 2 Digit: Student uses the distributive property to decompose numbers by place value into four partial products.
Developmental Multi-Digit Level 4	Student uses and understands expanded multiplication and division algorithms. For example: $25 \times 14 = 20 + 80 + 50 + 200 = 350$
Developmental Multi-Digit Level 5	Student uses and understands traditional multiplication and division algorithms. The student understands the place value of numbers and identifies the numbers as so. (Not carrying the one, but a ten or hundred, etc)